

# Equivalent Hamiltonian for Lee Model

H. F. Jones\*

Imperial College London

## Abstract

Using the techniques of quasi-Hermitian quantum mechanics and quantum field theory we use a similarity transformation to construct an equivalent Hermitian Hamiltonian for the Lee model. In the field theory confined to the  $V/N\theta$  sector it effectively decouples  $V$ , replacing the three-point interaction of the original Lee model by an additional mass term for the  $V$  particle and a four-point interaction between  $N$  and  $\theta$ . While the construction is originally motivated by the regime where the bare coupling becomes imaginary, leading to a ghost, it applies equally to the standard Hermitian regime where the bare coupling is real. In that case the similarity transformation becomes a unitary transformation.

## 1 Introduction

Following the original paper by Bender and Boettcher[1] on  $PT$ -symmetric but non-Hermitian Hamiltonians, subsequent research initially concentrated on an exploration of the reality or otherwise of the spectrum of non-Hermitian generalizations of well-known soluble models (see Ref. [2] for a systematic approach). However, it soon became apparent that something additional to the reality of the spectrum was needed if such models were to be viable as quantum theories with a proper probabilistic interpretation. This is because the natural metric for  $PT$ -symmetric models gives the overlap of two wave-functions  $\psi(x)$  and  $\varphi(x)$  as  $\int dx \varphi^*(-x)\psi(x)$ , rather than the usual  $\int dx \varphi^*(x)\psi(x)$ . Since the corresponding norm is not positive definite, the theory endowed with this metric does not represent a physical framework for quantum mechanics. Instead it turns out to be possible to construct[3, 4, 5]

---

\*h.f.jones@imperial.ac.uk

an alternative, positive-definite metric  $\eta \equiv e^{-Q}$ , which is dynamically determined by the particular Hamiltonian in question, satisfying

$$H^\dagger = e^{-Q} H e^Q. \quad (1)$$

$H$  is then said to be quasi-Hermitian with respect to  $\eta$ . It was also shown [6] that  $\rho = \sqrt{\eta}$  provided a similarity transformation from the non-Hermitian  $H$  to an equivalent Hermitian  $\tilde{H}$ , namely

$$\tilde{H} = e^{-Q/2} H e^{Q/2}, \quad (2)$$

and this equivalent Hermitian Hamiltonian was subsequently constructed, frequently in perturbation theory only, in a variety of models[7, 8, 9].

These techniques were used by Bender et al.[10] to show that the ghost state that appears in the Lee model[11, 12, 13] when the renormalized coupling constant exceeds a critical value can be treated as a viable state with a positive norm if the appropriate metric is used. Since they calculated this metric in the quantum mechanical version and in the  $V - N\theta$  sector of the field theory, a natural extension of their work is to attempt to calculate the equivalent Hermitian Hamiltonians in each case, obtained by the similarity transformation alluded to above. That is the task of the present paper.

In Section 2 we present the quantum mechanical calculation, which gives a closed-form expression for the equivalent Hermitian Hamiltonian. The field theory problem is tackled in Section 3, confined to the  $V - N\theta$  sector. We discuss the significance of these results in Section 4, noting in particular that the construction applies equally to the situation where the renormalized coupling is less than the critical value and the original Hamiltonian is also Hermitian.

## 2 Quantum Mechanics

The Hamiltonian for the simplified quantum-mechanical version of the Lee model is

$$H = H_0 + H_1, \quad (3)$$

where

$$\begin{aligned} H_0 &= m_{V_0} V^\dagger V + m_N N^\dagger N + m_\theta a^\dagger a, \\ H_1 &= g_0 (V^\dagger N a + a^\dagger N^\dagger V). \end{aligned} \quad (4)$$

Here  $N$  and  $V$  are treated as fermions, with occupation number 1 or 0. The bare states in the  $V/N\theta$  sector are denoted by  $|1, 0, 0\rangle$  and  $|0, 1, 1\rangle$ , where the indices are the occupation numbers of the bare  $V$ ,  $N$  and  $\theta$  particles respectively.

The energy eigenstates in the sector are given as linear combinations

$$|E\rangle = \alpha_1|1, 0, 0\rangle + \alpha_2|0, 1, 1\rangle, \quad (5)$$

with the coefficients  $\alpha_i$  given by the solutions of the secular equations

$$\begin{pmatrix} m_{V_0} - E & g_0 \\ g_0 & m_N + m_\theta - E \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0 \quad (6)$$

Introducing the variable

$$x \equiv \tan^{-1} \left( \frac{2g_0}{\mu_0} \right), \quad (7)$$

the two solutions can be written, for real  $g_0$ , as

$$\begin{aligned} E &\equiv m_V = m_{V_0} - g_0 \tan \frac{1}{2}x & \text{with } \alpha_1 &= \cos \frac{1}{2}x, \quad \alpha_2 = -\sin \frac{1}{2}x \\ \text{and} & & & \\ E &\equiv E_{N\theta} = m_N + m_\theta + g_0 \tan \frac{1}{2}x & \text{with } \alpha_1 &= \sin \frac{1}{2}x, \quad \alpha_2 = \cos \frac{1}{2}x \end{aligned} \quad (8)$$

Explicitly

$$g_0 \tan \frac{1}{2}x = \frac{1}{2} \left( \sqrt{\mu_0^2 + 4g_0^2} - \mu_0 \right). \quad (9)$$

In this notation the renormalized coupling constant is given by  $g = \mu \sin \frac{1}{2}x$ , where  $\mu \equiv m_N + m_\theta - m_V$ , and the relation between the bare and renormalized coupling constants by  $g = g_0 \cos \frac{1}{2}x$ , so that

$$g_0^2 = \frac{g^2}{1 - g^2/\mu^2}. \quad (10)$$

A similar equation appears in the field theory version, where a ghost appears when  $g$  exceeds a critical value and  $g_0$  becomes pure imaginary. However, contrary to what is stated in Ref. [10], the situation is rather different in quantum mechanics. Here the regime  $g > \mu$  is not viable, because it produces a complex spectrum.

Thus, as  $g$  passes through  $\mu$ , the unrenormalized coupling constant  $g_0$  goes to infinity and then becomes imaginary, with  $|g_0|$  very large. In that regime, assuming that  $\mu_0$  is real, it is clear from Eqs. (8) and (9) that  $m_V$

(and hence  $\mu$ ) is complex, contrary to supposition. The only loophole,  $\mu_0$  becoming complex, is ruled out by the equation

$$\mu_0 = \mu - \frac{g_0^2}{\mu}, \quad (11)$$

which can easily be derived from the above equations.

Nonetheless, we can proceed with the basic framework of Ref. [10], but staying within the allowed (Hermitian) regime  $g < \mu$ , so that  $g_0$  and  $x$  are both real. In that paper the equation for the operator  $Q$ , namely

$$[e^Q, H_0] = \{e^Q, H_1\}, \quad (12)$$

which is equivalent to the condition (1) for  $g_0$  imaginary, is solved, with the result that

$$Q = V^\dagger N a \frac{1}{\sqrt{n_\theta}} x_{n_\theta} - \frac{1}{\sqrt{n_\theta}} x_{n_\theta} a^\dagger N^\dagger V, \quad (13)$$

where, in analogy with Eq. (7) we have introduced the operator

$$x_{n_\theta} \equiv \tan^{-1} \left( \frac{2g_0 n_\theta}{\mu_0} \right). \quad (14)$$

Here  $n_\theta$  is the number operator for the  $\theta$  particle:  $n_\theta = a^\dagger a$ .

Because the  $V$  and  $N$  particles are treated as fermions, with the simplification that  $n_{V,N}^2 = n_{V,N}$  where  $n_V$  and  $n_N$  are the respective number operators, the series for  $e^Q$  can be summed exactly, with the result that, in our notation<sup>†</sup>

$$\begin{aligned} e^Q &= 1 - n_N(1 - n_V)(1 - \cos x_{n_\theta}) - n_V(1 - n_N)(1 - \cos x_{n_\theta+1}) \\ &\quad + V^\dagger N a \frac{1}{\sqrt{n_\theta}} \sin x_{n_\theta} - \sin x_{n_\theta} \frac{1}{\sqrt{n_\theta}} a^\dagger N^\dagger V, \end{aligned} \quad (15)$$

which can be usefully rewritten as

$$e^Q = 1 - f_1 n_N \bar{n}_V - f_2 n_V \bar{n}_N + V^\dagger N a f_3 - f_3 a^\dagger N^\dagger V, \quad (16)$$

where  $n_{V,N} \equiv 1 - n_{V,N}$  and

$$\begin{aligned} f_1 &= 1 - \cos x_{n_\theta}, \\ f_2 &= 1 - \cos x_{n_\theta+1}, \\ f_3 &= \frac{1}{\sqrt{n_\theta}} \sin x_{n_\theta}. \end{aligned} \quad (17)$$

---

<sup>†</sup>We have corrected a couple of errors in Eq. (32) of [10]

We have independently verified this result using the condition

$$H(-g_0) = e^{-Q} H(g_0) e^Q \quad (18)$$

rather than (12). In so doing, the identities

$$\begin{aligned} f_2 &= f_1(n_\theta + 1), \\ n_\theta f_3^2 &= 2f_1 - f_1^2, \\ f_3 &= (2g_0/\mu_0) \cos x_{n_\theta} \end{aligned} \quad (19)$$

proved crucial.

To construct the equivalent Hermitian Hamiltonian  $\tilde{H}$  we need  $e^{\frac{1}{2}Q}$ , which can be expressed as

$$e^{\frac{1}{2}Q} = 1 - \tilde{f}_1 n_N \bar{n}_V - \tilde{f}_2 n_V \bar{n}_N + V^\dagger N a \tilde{f}_3 - \tilde{f}_3 a^\dagger N^\dagger V, \quad (20)$$

where

$$\begin{aligned} \tilde{f}_1 &= 1 - \cos \frac{1}{2} x_{n_\theta}, \\ \tilde{f}_2 &= 1 - \cos \frac{1}{2} x_{n_{\theta+1}}, \\ \tilde{f}_3 &= \frac{1}{\sqrt{n_\theta}} \sin \frac{1}{2} x_{n_\theta}. \end{aligned} \quad (21)$$

The result, using identities similar to Eq. (19), is

$$\begin{aligned} \tilde{H} &= m_{V_0} n_V + m_N n_N + m_\theta n_\theta + g_0 \sqrt{n_\theta} \tan \frac{1}{2} x_{n_\theta} (n_N \bar{n}_V) \\ &\quad - g_0 \sqrt{n_\theta + 1} \tan \frac{1}{2} x_{n_{\theta+1}} (n_V \bar{n}_N). \end{aligned} \quad (22)$$

In the  $V/N\theta$  sector this reduces to

$$\tilde{H} \equiv m_{V_0} n_V + m_N n_N + m_\theta n_\theta + g_0 \tan \frac{1}{2} x (n_N n_\theta - n_V), \quad (23)$$

which decouples  $V$  from  $N$  and  $\theta$  and correctly reproduces the energy spectrum of Eq. (8). Note that because of the definition of  $x$  in Eq. (7),  $\tilde{H}$  is a function of  $g_0^2$  rather than  $g_0$  itself.

### 3 Field Theory in the $V/N\theta$ Sector

In field theory the free and interaction Hamiltonians are, respectively,

$$H_0 = \int d\mathbf{p} (m_{V_0} V_{\mathbf{p}}^\dagger V_{\mathbf{p}} + m_N N_{\mathbf{p}}^\dagger N_{\mathbf{p}}) + \int d\mathbf{k} \omega_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}},$$

(24)

$$H_1 = \int d\mathbf{p} d\mathbf{k} h_{\mathbf{k}} \left( V_{\mathbf{p}}^\dagger N_{\mathbf{p}-\mathbf{k}} a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger N_{\mathbf{p}-\mathbf{k}}^\dagger V_{\mathbf{p}} \right),$$

where  $\omega_{\mathbf{k}} = (m_\theta^2 + \mathbf{k}^2)^{\frac{1}{2}}$  and [10]

$$h_{\mathbf{k}} = \frac{g_0 \rho(\omega_{\mathbf{k}})}{(2\pi)^{3/2} (2\omega_{\mathbf{k}})^{1/2}}, \quad (25)$$

where  $\rho(\omega_{\mathbf{k}})$  is an appropriate cutoff.

As has been discussed in Ref. [10], it is again the case that above a certain critical value for the renormalized coupling constant the unrenormalized coupling constant  $g_0$  becomes imaginary. In field theory the spectrum remains real, but a ghost state appears, with negative residue, and the Hamiltonian becomes non-Hermitian (but  $\mathcal{PT}$ -symmetric).

Within the  $V/N\theta$  sector the  $Q$ -operator, satisfying (12), then has the form [10]

$$Q = \int d\mathbf{p} d\mathbf{k} \gamma_{\mathbf{k}} \left( V_{\mathbf{p}}^\dagger N_{\mathbf{p}-\mathbf{k}} a_{\mathbf{k}} - a_{\mathbf{k}}^\dagger N_{\mathbf{p}-\mathbf{k}}^\dagger V_{\mathbf{p}} \right), \quad (26)$$

where  $\gamma_{\mathbf{k}}$ , like  $h_{\mathbf{k}}$ , is imaginary. Again within the  $V/N\theta$  sector,  $Q^2$  turns out to be

$$Q^2 = \beta^2 \int d\mathbf{p} V_{\mathbf{p}}^\dagger V_{\mathbf{p}} - \int d\mathbf{p} d\mathbf{k}_1 d\mathbf{k}_2 \gamma_{\mathbf{k}_1} \gamma_{\mathbf{k}_2} a_{\mathbf{k}_1}^\dagger N_{\mathbf{p}-\mathbf{k}_1} N_{\mathbf{p}-\mathbf{k}_2} a_{\mathbf{k}_2}, \quad (27)$$

where  $\beta^2 \equiv - \int d\mathbf{k} \gamma_{\mathbf{k}}^2$ .

These are the only two independent structures, since it is easily seen that  $Q^3 = \beta^2 Q$  within the  $V/N\theta$  sector, i.e. ignoring terms that vanish when acting on a state in this sector. Thus the series for  $e^Q$  can be readily summed, to give

$$e^Q = 1 + c_1 Q + c_2 Q^2. \quad (28)$$

where

$$\begin{aligned} c_1 &= \frac{\sinh \beta}{\beta}, \\ c_2 &= \frac{\cosh \beta - 1}{\beta^2}, \end{aligned} \quad (29)$$

satisfying the important identity

$$c_1^2 = 2c_2 + c_2^2 \beta^2. \quad (30)$$

Inserting the above expression for  $e^Q$  into Eq. (12) yields the condition

$$c_1 \mu_{\mathbf{k}} \gamma_{\mathbf{k}} = c_2 \beta_2 \gamma_{\mathbf{k}} + \frac{c_1^2}{c_2} h_{\mathbf{k}}, \quad (31)$$

where  $\mu_{\mathbf{k}} = \omega_{\mathbf{k}} + m_N - m_{V_0}$  and  $\beta_2 \equiv -\int d\mathbf{k} \gamma_{\mathbf{k}} h_{\mathbf{k}}$ . The other condition, that  $\gamma_{\mathbf{k}}$  be imaginary, has already been assumed. In principle  $\gamma_{\mathbf{k}}$  is found from this equation together with the defining relations for  $\beta^2$  and  $\beta_2$ .

We have again verified these calculations using the condition (1) rather than (12). This amounts to verifying the relation

$$\begin{aligned} 2H_1 = & c_1 [Q, H] + c_1^2 Q H Q - c_2 \{Q^2, H\} \\ & - c_1 c_2 (Q^2 H Q - Q H Q^2) - c_2^2 Q^2 H Q^2 \end{aligned} \quad (32)$$

and equating coefficients of the various operators that arise.

There are in fact just three of these, namely

$$\begin{aligned} \mathcal{O}_1 &= V_{\mathbf{p}}^\dagger V_{\mathbf{p}}, \\ \mathcal{O}_2 &= a_{\mathbf{k}_1}^\dagger N_{\mathbf{p}-\mathbf{k}_1} N_{\mathbf{p}-\mathbf{k}_2}^\dagger a_{\mathbf{k}_2}, \\ \mathcal{O}_3 &= V_{\mathbf{p}}^\dagger N_{\mathbf{p}-\mathbf{k}} a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger N_{\mathbf{p}-\mathbf{k}}^\dagger V_{\mathbf{p}}. \end{aligned} \quad (33)$$

Thus in the expression on the RHS we require that the coefficients of  $\mathcal{O}_1$  and  $\mathcal{O}_2$  vanish, and that that of  $\mathcal{O}_3$  match the coefficient in  $2H_1$ . An important ingredient in the manipulations is an identity obtained by integrating Eq. (31) with respect to  $\int d\mathbf{k} \gamma_{\mathbf{k}}$ , namely

$$c_1 \int d\mathbf{k} \gamma_{\mathbf{k}}^2 \mu_{\mathbf{k}} = -\beta_2 \left( c_2 \beta^2 + \frac{c_1^2}{c_2} \right). \quad (34)$$

To construct the equivalent Hermitian Hamiltonian  $\tilde{H}$  we need  $e^{\frac{1}{2}Q}$ , which in this case reads

$$e^{\frac{1}{2}Q} = 1 + \tilde{c}_1 Q + \tilde{c}_2 Q^2, \quad (35)$$

where

$$\begin{aligned} \tilde{c}_1 &= \frac{\sinh \frac{1}{2}\beta}{\beta}, \\ \tilde{c}_2 &= \frac{\cosh \frac{1}{2}\beta - 1}{\beta^2}, \end{aligned} \quad (36)$$

satisfying

$$\tilde{c}_1 c_1 - \beta^2 \tilde{c}_2 c_2 = c_2. \quad (37)$$

Again only the three structures  $\mathcal{O}_1$ ,  $\mathcal{O}_2$  and  $\mathcal{O}_3$  are produced, but this time the coefficient of  $\mathcal{O}_3$  vanishes, leaving

$$\begin{aligned}\tilde{H} = & H_0 + \frac{\beta_2}{\beta} \tanh \frac{1}{2}\beta \int d\mathbf{p} V_{\mathbf{p}}^\dagger V_{\mathbf{p}} \\ & + 2 \frac{\cosh \frac{1}{2}\beta - 1}{\beta \sinh \beta} \int d\mathbf{p} d\mathbf{k}_1 d\mathbf{k}_2 a_{\mathbf{k}_1}^\dagger N_{\mathbf{p}-\mathbf{k}_1} N_{\mathbf{p}-\mathbf{k}_2}^\dagger a_{\mathbf{k}_2} \times \\ & \times [\cosh \frac{1}{2}\beta (\gamma_{\mathbf{k}_1} h_{\mathbf{k}_2} + \gamma_{\mathbf{k}_2} h_{\mathbf{k}_1}) - \beta_2 \tilde{c}_2 \gamma_{\mathbf{k}_1} \gamma_{\mathbf{k}_2}] \end{aligned} \quad (38)$$

We will make a more detailed comparison with the corresponding quantum-mechanical result, Eq. (22), in the next section, but the general features are clear: the  $V$  and  $N\theta$  sectors are again separated, with a term that renormalizes the mass of the  $V$  particle together with a four-point interaction between  $N$  and  $\theta$ .

## 4 Discussion

We can find the quantum-mechanical limit of Eq. (38) by ignoring all momentum subscripts and integrals and identifying  $\gamma = x$ , by comparison of the two expressions for  $Q$ , Eqs. (13) and (26). By comparison of the two expressions for  $H_1$ , Eqs. (4) and Eqs. (24), we further identify  $h = g_0$ . Recall that the quantum-mechanical result is only valid for  $g_0$  real. Thus we have to continue the field-theory expressions to this regime, where  $\gamma$  is real. Thus  $\beta^2 = -x^2$ , so that  $\beta = \pm ix$ , while  $\beta_2 = -g_0 x$ . It follows almost immediately that the coefficient of  $V^\dagger V$  reduces correctly to  $-g_0 \tan \frac{1}{2}x$ , while the coefficient,  $+g_0 \tan \frac{1}{2}x$ , of  $n_N n_\theta$  follows after a little algebra.

Perhaps the most important feature of our work is that we can use Eqs. (12) or (18) for both real and imaginary  $g_0$ . In the case when  $g_0$  is imaginary and  $H$  is non-Hermitian, this is equivalent to the usual equation for quasi-Hermiticity, Eq. (1). Then  $Q$  is Hermitian so that the relation (2) represents a similarity transformation rather than a unitary transformation, and consequently the metric of Hilbert space is changed. However, in the case when  $g_0$  is real, and the original  $H$  is actually Hermitian, we can still derive a  $Q$  operator from Eqs. (12) or (18). In this case  $Q$  is anti-Hermitian and the relation (2) is a unitary transformation relating one Hermitian Hamiltonian to another, with no change of the standard Hilbert space metric<sup>‡</sup>. The common feature of both cases is that, whereas the initial Hamiltonian is linear

---

<sup>‡</sup>The metric operator  $\eta$  is only equal to  $e^{-Q}$  for  $Q$  Hermitian. The general relation is  $\eta = e^{-\frac{1}{2}Q^\dagger} e^{-\frac{1}{2}Q}$ , which instead reduces to the identity for  $Q$  anti-Hermitian.



in  $g_0$ , the equivalent Hamiltonian is a function of  $g_0^2$ . Thus, using a method inspired by the study of pseudo-Hermitian Hamiltonians, we have produced a Hermitian four-point interaction Hamiltonian equivalent to the original Hermitian trilinear interaction Hamiltonian. This equivalent Hamiltonian  $\tilde{H}$  is given in Eq. (38), where  $\beta$  is real in the ghost regime, but pure imaginary in the Hermitian regime of  $H$ .

In graphical terms it is not obvious how  $\tilde{H}$  is constructed. To order  $g_0^2$  it amounts to contracting the  $V$  propagator in the  $N\theta$  scattering amplitude to a point, but it also takes into account higher-order loop diagrams in a complicated way.

## References

- [1] C. M. Bender and S. Boettcher, Phys. Rev. Lett., **80** (1998) 5243.
- [2] G. Levai and M. Znojil, J. Phys. A: Math. Gen. **33** (2000) 7165.
- [3] C. M. Bender, D. C. Brody and H. F. Jones, Phys. Rev. Lett. **89** (2002) 270401; Erratum *ibid.* **92** (2004) 119902.
- [4] F. Scholtz, H. Geyer and F. Hahne, Ann. Phys. **213** (1992) 74.
- [5] A. Mostafazadeh, J. Math. Phys. **43** (2002) 205.
- [6] A. Mostafazadeh, J. Phys. A: Math. Gen. **36** (2003) 7081.
- [7] H. F. Jones, J. Phys. A: Math. Gen. **38** (2005) 1741.
- [8] A. Mostafazadeh, J. Phys. A: Math. Gen. **38** (2005) 6557; Erratum *ibid.* 8185.
- [9] H. F. Jones and J. Mateo, Phys. Rev. **D73** (2006) 085002.
- [10] C. M. Bender, S. F. Brandt, J.-H. Chen and Q. Wang, Phys. Rev. D **71** (2005) 025014.
- [11] T. D. Lee, Phys. Rev. **95** (1954) 1329.
- [12] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper & Row, New York, 1962), Chap. 12
- [13] G. Barton, *Introduction to Advanced Theory* (John Wiley & Sons, New York, 1963), Chap. 12